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## EFFECTIVE TRANSVERSE CONDUCTIVITY OF LAYERED MATERIALS WITH THROUGH CRACKS IN THE LAYERS

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The problem of the transverse conductivity of a multilayered packet when loaded in the same plane as the layers, which are mechanically independent, in the presence of or on the generation of a number of through cracks in the layers under load is analyzed.

1. For the sake of brevity we shall refer in this and the subsequent sections to thermal conductivity, but diffusion taking into account the influence of changes in concentration on changes in volume can be treated analogously.

1.1. Let us consider a packet of layers, parallel with the  $xy$  plane, containing a number of through cracks (see Fig. 1). The conductivity of unloaded layers with cracks will be treated as negligible compared with the conductivity of the medium in the exposed cracks and of the medium between the layers; the mean resistance of the medium between the layers will be neglected. The loading of the layers is determined by the conditions at the periphery of the packet. When the cracks are exposed by the action of extension, the layer acquires a finite effective conductivity and under certain conditions leakage may occur through the packet, i.e., it loses its insulating properties. A similar situation arises in many cases of practical importance. The problem of the loss of insulating properties in a multilayered packet is examined theoretically below.

The effective conductivity of the packet perpendicular to the layers at a certain point  $z$  under the conditions indicated is equal to

$$\lambda = \lambda' \sum_a S^{(a)}. \quad (1)$$

Here  $S^{(a)}$  is the area of the exposed  $a$  crack (a jump in the normal component of displacement integrated along the length of the crack) and  $\lambda'$  is the coefficient of thermal conductivity of the medium in the cracks. Assuming that the layers with cracks are under plane stressed state conditions, we have (for derivation see [1, 2])

$$\varepsilon_{ik} = \varepsilon_{ik}^0 + \frac{1}{2} \sum_a (n_i S_k^{(a)} + n_k S_i^{(a)}). \quad (2)$$

Here  $\varepsilon_{ik}$  is the effective strain tensor of the layer,  $\varepsilon_{ik}^0$  is the mean strain tensor of the material outside the crack (the strains are assumed to be slight),  $n_i$  is the vector of the unit line normal to the crack (the cracks are assumed to be rectilinear), and  $S_i^{(a)}$  is the  $i$ -component of the displacement jump integrated along the length of the crack. Since  $n_i S_i^{(a)} = S^{(a)}$  (recurrent index summing), it follows from (1) and (2) that

$$\lambda = \lambda' (\varepsilon_{ii} - \varepsilon_{ii}^0). \quad (3)$$

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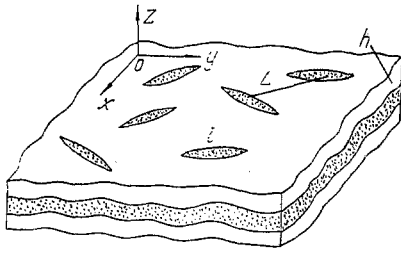


Fig. 1. Diagram of packet of layers. Dots indicate the medium in cracks and between layers which is much more conducting than the material of which the layers are made. Flow parallel with z axis. Layers do not interact mechanically. The packet is loaded in xy plane.

For this expression to be applicable the temperature equalization must proceed much more rapidly between the layers than across the packet and for these purposes the crack exposure must not be too great. On the other hand, the crack exposure should not be too small either, since it is considered possible to neglect the conductivity of the unloaded layers. Under conditions in which the thermal properties of the medium in the cracks and between the layers, as well as the thicknesses  $h$  of the layers and the spaces between the layers, are of the same order of magnitude, which are easily satisfied, the corresponding evaluations take the form

$$\frac{\lambda_m}{\lambda'} \ll \varepsilon_{ii} - \varepsilon_{ii}^0 \approx \frac{h^2}{L^2}, \quad (4)$$

where  $\lambda_m$  is the coefficient of thermal conductivity of the layer material and  $L$  is the mean distance between cracks. It is clear from this that the crack density should not be too small.

According to (3), the effective transverse conductivity  $\lambda$  is expressed by the product of a large magnitude  $\lambda'$  multiplied by a small magnitude  $\varepsilon_{ii} - \varepsilon_{ii}^0$  and may therefore vary over a wide range of values. It should be noted that in addition to cracks, the above also refers to crack-like formations with a finite mechanical resistance under conditions in which the conductivity is fairly high.

1.2. The strains in (3) are related to the tensor of stresses  $\sigma_{ik}$  and the mechanical characteristics of the material. For isotropic layers with an isotropic crack distribution the relations of thermoelasticity [3] take the form

$$\varepsilon_z^0 = E_0^{-1}(\sigma_x - \nu_0 \sigma_y) + \alpha \theta, \quad \varepsilon_x = E^{-1}(\sigma_x - \nu \sigma_y) + \alpha \theta \quad (5)$$

(and analogously for  $\varepsilon_y^0$  and  $\varepsilon_y$ ). Here  $\sigma_x$  and  $\sigma_y$  are the mean stresses,  $E_0$  and  $\nu_0$  are Young's modulus and Poisson's coefficient for the layer material,  $E$  and  $\nu$  are the corresponding effective characteristics,  $\theta$  is the temperature deviation, and  $\alpha$  is the coefficient of linear expansion, which obviously is the same in both cases.

If the system of cracks is not isotropic, the second correlation in (5) varies. Thus, for cracks parallel with the x axis we have

$$\varepsilon_x = \varepsilon_x^0 = E_0^{-1}(\sigma_x - \nu_0 \sigma_y) + \alpha \theta, \quad \varepsilon_y = E_*^{-1}(\sigma_y - \nu_* \sigma_x) + \alpha \theta, \quad (6)$$

in which  $\nu_0 E_0^{-1} = \nu_* E_*^{-1}$ .

1.3. If the cracks are not drawn too close together, so that interaction between them is not significant,  $E$  and  $\nu$  and  $E_*$  and  $\nu_*$  can be found from the solution for one crack in an infinite body (the low-concentration approximation). These characteristics can also be found from the solution to this equation when the interaction is significant if the distribution of cracks by size is fairly broad and / or if the cracks are distributed through space fairly evenly and uniformly. These questions and the results of an experimental test are examined in [1, 2]. The corresponding formulas (the same regardless of whether there is a broad or narrow distribution by size but with a fairly random crack distribution through space) for an isotropic material take the form

$$E = E_0 \exp(-\pi\Omega/4), \quad \nu E^{-1} = \nu_0 E_0^{-1} \quad (7)$$

in the case of an isotropic crack distribution [1], and

$$E_* = 9E_0 [(3\pi\Omega/16 + 2)^2 - 1]^{-2}, \quad \nu_* E_*^{-1} = \nu_0 E_0^{-1} \quad (8)$$

in the case of cracks parallel with the x axis. Here  $\Omega = \sum N_j l_j^2 = N \bar{l}^2$  with  $N_j$  being the number of cracks of length  $l_j$  per unit area and  $N = \sum N_j$  (formulas (8) are obtained as indicated in [1] using the solution for a crack in an anisotropic plate [4, 5]). By combining (5) and (7) or (6) and (8) it is possible to obtain an expression for  $\lambda$  in terms of stresses, temperature, and crack characteristics.

1.4. The stressed state influencing  $\lambda$  may be created both by the application of mechanical loads and by changes in temperature. For example,  $\lambda$  is negligibly small for a packet secured along a circular periphery with an isotropic cracked state when heated ( $\theta > 0$ ) and it can be varied considerably when the packet is cooled

as a result of biaxial extension. By making use of (5), (7), and (3) and assuming that  $\varepsilon_x = \varepsilon_y = 0$ , we find

$$\lambda = -\lambda' \varphi(\Omega) \alpha \theta H(\theta), \quad \varphi(\Omega) = 2 \frac{1 - \exp(-\pi\Omega/4)}{1 - \nu_0 \exp(-\pi\Omega/4)}, \quad (9)$$

where  $H(\theta) = 0$  when  $\theta < 0$  and  $H(\theta) = 1$  when  $\theta > 0$ . Analogously, for a packet with a distribution of cracks parallel with the x axis and secured along the edges parallel with the cracks ( $\varepsilon_y = 0$ ) and free of stresses in the direction along the cracks ( $\sigma_x = 0$ ), we find from (6), (8), and (3) that

$$\lambda = -\lambda' \varphi(\Omega) \alpha \theta H(\theta), \quad \varphi(\Omega) = 1 - 9[(3\pi\Omega/16 + 2)^2 - 1]^{-2}. \quad (10)$$

2. In the previous section the characteristics of the cracked state are assumed to be predetermined. Let us here examine the question of the growth of the cracked state. The basic principles will be valid to an equal extent for both the plane and spatial cases.

2.1. The term "growth of cracked state" is taken to mean two processes: the process of crack initiation and the process of crack growth. This division is to a certain extent arbitrary. If the term crack is taken to mean, as is usual, the fracture surface of displacements in an elastic body such that the actual failure which causes the crack growth and, possibly, the inelastic strain effects preceding failure are concentrated in small regions at the ends of the crack (end regions), then the first of the two processes is characterized by a change in the density  $N_0$  of cracks of the minimum length consistent with such a concept and the second is characterized by a change in  $N_j$  ( $j > 0$ ) as a result of increasing the lengths  $l_j$ .

Cracks of length  $l_0$ , generated in the time interval from  $t'$  to  $t' + dt'$  in quantities of  $\dot{N}_0(t') dt'$  (per unit area), achieve lengths of  $l(t, t')$  at a certain moment  $t$ . Therefore, for  $\Omega$  [examples (9) and (10)] we have

$$\Omega(t) = \int_0^t \dot{N}_0(t') l^2(t, t') dt'. \quad (11)$$

The  $N_0$  and  $l$  functions are, in principle, known if the law governing the accumulation of cracks of length  $l_0$  and the law governing the growth of cracks are known. Cracks accumulate if the rate of the first process is greater than that of the second. In particular, for conditions in which a crack density of  $N_0 = \text{const}$ , such that  $N_0 = N_0 \delta(t - t_0)$ , emerges as soon as a predetermined stress  $\sigma_0$  is reached at a moment  $t_0$ , it follows from (11) that

$$\Omega(t) = N_0 l^2(t, t_0) \quad (12)$$

(if cracks exist from the very beginning it should be assumed here that  $t_0 = 0$ ). In general, if at a moment  $t_0$  a crack distribution by lengths with a density  $n(l)$  exists or is generated and if thereafter only those cracks grow without any new cracks emerging, then

$$\Omega(t) = \int n(l') l^2(t, t_0, l') dl', \quad (13)$$

where in addition to  $t_0$ ,  $l'$  is now also an argument of  $l^2$ . Under totally general conditions  $l$  in (12) and (13) can be found in a quasistationary approximation in which the instantaneous velocity of the end of the crack is dependent only on the coefficient of intensity of the stresses  $N_e$  at a given crack end at the same moment in time (see [6]). This approximation takes a particularly simple form when the low-concentration approximation is also valid at the same time. Experiment shows that this remains satisfactory up to  $\Omega \approx 0.3$  [2]. In the low-concentration approximation  $\Omega$  should be treated as small and only those terms which are linear in terms of  $\Omega$  should be retained. Then in (9)  $\varphi = \pi\Omega/2(1 - \nu_0)$ , in (10)  $\varphi = \pi\Omega/2$ , and the equation for  $l$  takes the form

$$\frac{dl}{dt} = f(N_e, \theta). \quad (14)$$

Here  $f$  is the material function and  $N_e = 1/2\sigma\sqrt{l}$ , where  $\sigma$  is the stress. For example (9) and (10)  $\sigma = -E'\alpha\theta(1 - c\Omega)$  while in (9)  $E' = E_0/(1 - \nu_0)$ ,  $c = \pi/4(1 - \nu_0)$  and in (10)  $E' = E_0$ ,  $c = \pi/2$ . By inserting all this into (14) we obtain an equation for finding  $l(t, t', l')$ . As a result and taking into account (13), (9), and (10),  $\lambda$  is expressed as a functional from  $\theta$ .

2.2. Let us now assume that macroscopic structural components (inclusions, grains, etc.) are present in the material and let us also examine macroscopic cracks of substructural size. Such cracks can accumulate as a result of stopping at inclusions and they may accumulate at the grain boundaries or at the matrix-inclusion interfaces. If the process of crack initiation and propagation to a size  $l_*$ , governed by the mean distance between inclusions or by grain size, is gradual in time, then the examination can be carried out as envisaged in [1]. If the time effects are not significant, energetic principles analogous to Griffiths' [8] can be used and in this case it is assumed that the same energy  $\zeta$  is consumed in the growth of each crack in a layer of unit

thickness. Then an energy of  $N\xi = 2N\gamma l_*$  is consumed in the growth of  $N$  cracks, where  $\gamma$  has the sense of the averaged specific failure energy. These consumptions should be compensated by a change in the density of the strain energy. For example, in the cases of loading examined above  $W = E(\alpha\theta)^2/(1-\nu)$  for (9) and  $W = E_* \times (\alpha\theta)^2/2$  for (10). Crack accumulation proceeds until

$$\Phi = W(\Omega) + 2N\gamma l_*, \quad \Omega = Nl_*^2; \quad (15)$$

it falls with the growth of  $N$ . An equilibrium value of  $N_m$  under the given conditions corresponds to a minimum  $\Phi$ .

For example (9) [example (10) is analyzed analogously] the condition for a minimum  $\Phi$  takes the form

$$\frac{\xi}{(1-\nu_0\xi)^2} = \eta; \quad \xi = \exp(-\pi\Omega/4), \quad \eta = \frac{8\gamma}{\pi(\alpha\theta)^2 E_* l_*} \quad (16)$$

The minimum when  $N = N_m$  exists if

$$\eta < \frac{1}{(1-\nu_0)^2} = \eta_0 \text{ или } \theta < \theta_0 = - \sqrt{\frac{8\gamma(1-\nu_0)^2}{\pi\alpha^2 E_* l_*}} \quad (17)$$

Then

$$N_m = -\frac{4}{\pi l_*} \ln \xi_m; \quad \xi_m = \frac{2\nu_0\eta + 1 - \sqrt{4\nu_0\eta + 1}}{2\nu_0^2\eta} \quad (18)$$

Taking into account (15) and (17), we now have instead of (9)

$$\lambda = -2\lambda'\alpha\theta \frac{\sqrt{4\nu_0\eta + 1} - 1 - 2\nu_0(1-\nu_0)\eta}{\nu_0(\sqrt{4\nu_0\eta + 1} - 1)} H(\eta_0 - \eta) \quad (19)$$

Such a calculation is valid if there is no barrier for crack initiation. If there is a barrier, the transition from  $N = 0$  to  $N = N_m(\theta)$  occurs when  $\theta < \theta' < \theta_0 < 0$  so that in (19)  $\eta_0$  should be replaced by  $\eta' < \eta$ . Here  $\theta'$  corresponds to the stress  $\sigma_b$  at which the barriers are removed (the introduction of a threshold  $\sigma_0$  assumes a clearly defined relationship between the initiation barriers and the stress). The  $N_m$  value is restricted from above either by the condition for exhausting the possibilities of crack initiation or by the condition for matrix failure after which crack growth and then crack coalescence begin.

3. Let us examine the influence of the cracked state on the transfer processes.

3.1. The equation for  $\theta$  neglecting the influence of the stressed state on the liberation (absorption) of heat or of the diffusing substance takes the form

$$\bar{\rho}c \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z} \left( \lambda \frac{\partial\theta}{\partial z} \right) \quad (20)$$

Here  $\bar{\rho}c$  is the product of the density multiplied by the thermal capacity averaged for a fairly large number of layers [ $\bar{\rho}c = (\rho c)_1\beta + (\rho c)_2(1-\beta)$ ] (where the indices 1 and 2 denote averaging by layers and by the interlayer medium, respectively, and  $\beta$  is the portion of the thickness of the packet taken up by the layers) and  $\lambda$  is the function or functional from  $\theta$  and may also be dependent on the stresses applied which are not related to  $\theta$ . In the case of diffusion  $\bar{\rho}c = 1$ ,  $\lambda$  has the sense of the effective coefficient of diffusion,  $\theta$  is the change in concentration, and  $\alpha$  is the coefficient of shrinkage. The processes of thermal conductivity and diffusion can proceed together influencing each other through the cracked state. Such will, for example, be the case when the coefficient of diffusion is determined by the cracked state, controlled by the change in temperature, and when the shrinkage coefficient, related to the change in concentration, and the influence of the cracked state on the thermal characteristics are negligibly low. In this case we have a system of two equations in the form of (20) for  $\theta_c$  and  $\theta_T$  with the coefficient  $\lambda$  in the first being dependent on  $z$  and  $t$  and in the second being constant.

3.2. When the cracked state is fixed or changes fairly slowly the flow  $q$  through the packet can be found, neglecting the time derivative in (20). Taking into account that  $q = -\lambda\partial\theta/\partial z$  in cases (9) and (10) when  $\theta < 0$  we find

$$q = \frac{1}{2} \lambda' \alpha [\theta^2(0, t) - \theta^2(H, t)] \left[ \int_0^H \frac{dz}{\varphi(\Omega(z, t))} \right]^{-1}, \quad (21)$$

where  $H$  is the thickness of the packet with  $\Omega$  dependent on  $z$  and  $t$ . Equation (21) can be used to find the leakage through the packet if the values for  $\theta$  on both sides of it are known. In the general case (21) is a boundary

condition at that point of the boundary occupied by the cracked packet in the corresponding boundary-value problem. Given processes of diffusion and temperature change related through the cracked state, if the thermal conductivity is assumed to be so great that in the case examined above  $\theta_T$  can be treated as constant through the thickness of the packet, formula (21) is obtained for the flow of the diffusing substance by replacing  $1/2\alpha[\theta^2(0, t) - \theta^2(H, t)]$  by  $\alpha\theta_T[\theta_c(0, t) - \theta_c(H, t)]$  in it.

3.3. The time from the moment at which leakage through the packet begins can be determined from non-stationary solutions to Eq. (20) or to a corresponding system. Let, for example, drying occur accompanied by shrinkage in a fixed cracked state not dependent on  $z$  and with an instantaneous reduction in concentration to a value of  $-\theta_c$  at the moment  $t = 0$  at the boundary of the packet after which the value remains constant. It can then be considered that the loading conditions correspond to examples (9) and (10). Due to the nonlinearity of the problem the rate  $\nu$  of propagation of the concentration reduction wave is finite (for an analysis of this kind of problem and the solution see [9]). In the given case [9] (IV.1.40)

$$\nu = 0.81 \sqrt{\frac{\lambda' \alpha \theta_c}{\varphi}} t_*^{-1}, \quad t_* = 0.38 H^2 [\lambda' \alpha \theta_c \varphi(\Omega)]^{-1}, \quad (22)$$

where  $t_*$  is the time taken for the front to pass through a packet of thickness  $H$ .

For an initially undamaged packet the condition for the generation of a cracked state should be taken into account. For the mechanism examined in Sec. 2.2 and a load corresponding to example (9), we have an equation of type (20) in which  $\lambda$  is governed by formula (19). It should be noted that  $\gamma$  can be heavily dependent on  $\theta$ . The generation of a cracked state and with it of a  $\theta$  reduction wave occurs only when  $\theta$  falls at the boundary of the packet below a value  $\theta_0$  governed by Eq. (17) (or the corresponding value  $\theta'$  if there is a crack initiation barrier). A  $z$ -dependent cracked state is generated when the concentration reduction wave passes, in which the form of the relationship  $\Omega(z)$  (found from the formulas in Sec. 2.2) and the depth to which the cracked state penetrates are governed by the nature and duration of the fall in  $\theta$  at the packet boundary. When the effect is repeated the propagation of the  $\theta$  reduction wave in the region of the incipient cracked state proceeds as described above. The cracked state front advances further (if it has not passed right through the thickness of the packet in the first instance) only when  $\theta$  is reduced sufficiently at the site where the front had stopped previously.

The examples considered show how (if the mechanical characteristics of the layers, including the resistance to crack generation and crack growth which determine them, are known and if the conductivity of the medium in the cracks and between the layers, which must greatly exceed the conductivity of the layers, is also known) it is possible in the presence of various different forms of external influences to establish the feasibility and nature of the loss by a packet of layers of its insulating properties, as well as to establish leakage through the disrupted packet and the time it will take to happen and during which it will remain significant.

#### NOTATION

$\lambda'$ ,  $\lambda_m$ ,  $\lambda$ , conductivity of medium in cracks, unloaded layers, and packet (transverse, effective), respectively;  $S^{(a)}$ , area of exposed crack;  $\varepsilon_{ik}$ , tensor of effective infinitely small strains ( $i = x, y; k = x, y$ );  $\varepsilon_{ik}^0$ , averaged tensor of elastic infinitely small strains between cracks;  $n_i$ , vector of unit line normal to crack line;  $S_i^a$ , components of displacement jump on crack, integrated along length of crack;  $h$ , thickness of layer (it is assumed that the distance between the layers is of the same order of magnitude);  $L$ , mean distance between cracks;  $\sigma_{ik}$ , stress tensor;  $\theta$ , temperature (concentration) difference;  $\alpha$ , coefficient of linear temperature (concentrational) expansion;  $E_0$ ,  $\nu_0$ , Young's modulus and Poisson's coefficient for layer material;  $E$ ,  $\nu$  the same, for effective characteristics in isotropic case;  $E_*$ ,  $\nu_*$ , the same, for effective characteristics when cracks are parallel;  $N_j$ ,  $N$ ,  $N_m$ , numbers of cracks per unit area;  $l_j$ ,  $l$ ,  $l_*$ , lengths of crack;  $\Omega$ , dimensionless crack concentration;  $H(s)$  function of a single crack;  $f$ , material function governing crack-propagation resistance;  $N_\varepsilon$  coefficient of stress intensity;  $\gamma$ , mean specific failure energy;  $W$ , strain energy density;  $\rho c$ , averaged product of density multiplied by thermal capacity;  $(\rho c)_1$ , the same, for layers;  $(\rho c)_2$ , the same, for medium between layers;  $\beta$ , part of packet thickness occupied by layers;  $\theta_c$ ,  $\theta_T$ , concentration and temperature difference, respectively;  $H$  packet thickness;  $q$ , flow (of heat or substance) through packet.

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## THERMAL RESISTANCE OF A SYSTEM OF PARALLELEPIPEDS

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UDC 536.248.1

A method is proposed for calculating the thermal resistance of a system of parallelepipeds with a local source which is encountered in the analysis of the thermal conditions in hybrid integral microcircuits.

Many applied problems, particularly problems associated with the analysis of thermal conditions in microelectronic devices, reduce to a thermal model which is a "pyramid" of  $n$  unlike parallelepipeds of different sizes (Fig. 1a). In the actual construction, the parallelepipeds forming the pyramid are the backing, "chips," adhesive layer, switching plate, base, etc. [1]. There are rectangular, flat energy sources on the upper surface of the first parallelepiped. Thermal contact between adjacent surfaces is assumed ideal. Heat dissipation from the lower surface of the  $n$ -th parallelepiped obeys Newton's law and is characterized by a heat-exchange coefficient  $\alpha$ ; there is no heat exchange at the lower surfaces.

The exact mathematical description of the temperature field in such a system is rather complex and hardly can be used for practical purposes. Calculation of the thermal resistance from the source to the environment is usually based on the construction of an equivalent circuit representing a chain of series-connected thermal resistances [1-4]. It is further assumed that the interfaces are isothermal.

We analyzed the possibility of such an approach for the following problem: a bounded cylinder with a local energy source on one end and boundary conditions of the first and third kind on the opposite end. The thermal resistance from the source to the environment was calculated in the two cases. A comparison of the results showed that the values of the thermal resistance can differ by almost a factor of two for given values of the Biot number and given ratios of cylinder and source sizes. Therefore, determination of the thermal resistance of this system must be carried out with consideration of heat-transfer conditions at the heat-releasing surfaces of each body. A method is proposed below for which sequential application provides an accuracy sufficient for practical purposes without significant complication of the computational process.

### Method for Determination of Thermal Resistance

We shall show that for this system of bodies, the problem of determining the total thermal resistance from energy source to environment can be reduced to a problem of determining the thermal resistance of the first parallelepiped, the heat-transfer conditions at the lower boundary of which are characterized by an equivalent coefficient  $\alpha_1$  that includes the effect of all the other parallelepipeds (Fig. 1c).

To determine the value of  $\alpha_1$ , we consider successively the temperature field of each  $i$ -th parallelepiped, for which the thermal model can be represented in the following manner: on the upper face of the parallelepiped, there is a flat energy source with an intensity  $p$  and area  $S_{i-1}$ ; on the lower surface, heat transfer is characterized by a heat-transfer coefficient  $\alpha_i$ , which takes into account the effect of all the remaining  $(n - i)$

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